

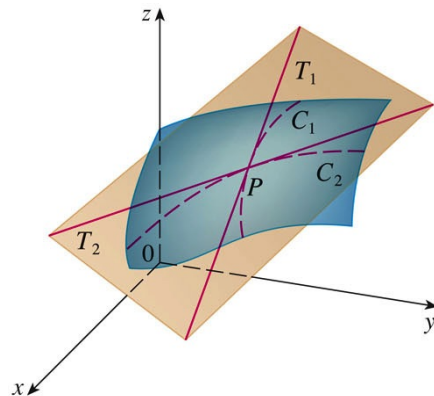
Lesson 14. Tangent Planes and Linear Approximations

0 Warm up

Example 1. Find an equation of the plane that passes through $(-1, 1, 5)$ and is perpendicular to the vector $\langle 2, 4, -3 \rangle$.

1 Tangent planes

- Let S be a surface with equation $z = f(x, y)$
- Let $P(x_0, y_0, z_0)$ be a point on S
 - So $z_0 =$
- Let T_1 and T_2 be the tangent lines at P in the x - and y -directions, respectively
- The **tangent plane** to the surface S at point P is the plane that contains both tangent lines T_1 and T_2



- The tangent plane must have an equation of the form $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or equivalently,

- If this is the equation of the tangent plane, its intersection with the plane $y = y_0$ must be the tangent line T_1

- Setting $y = y_0$, we obtain

- Looking at this equation A must be

- Similarly, B must be

⇒ An equation of the tangent plane to the surface $z = f(x, y)$ at point $P(x_0, y_0, z_0)$ is

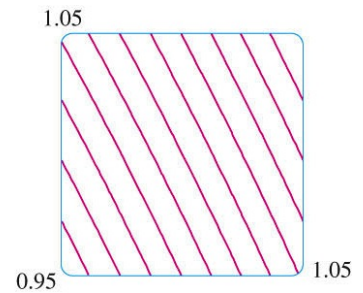
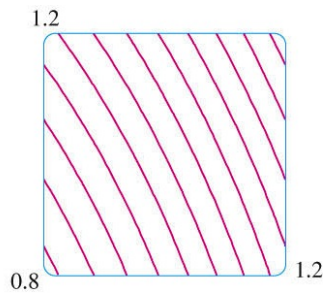
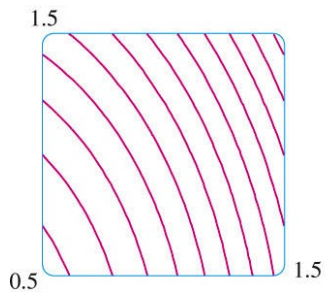
Example 2. Find the tangent plane to the surface $z = x^2 + xy + 3y^2$ at the point $(1, 1, 5)$.

2 Linear approximations

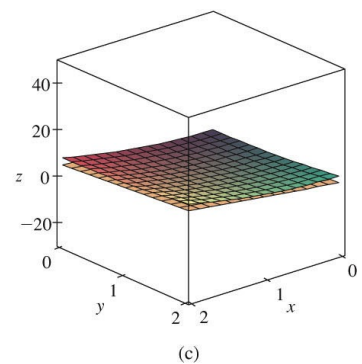
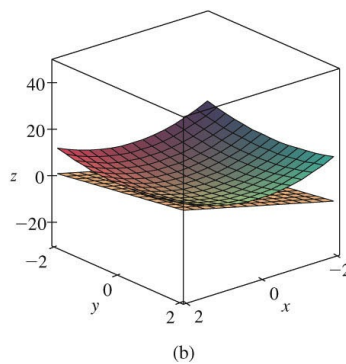
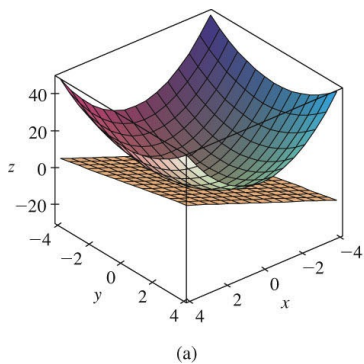
- What do the level curves of a plane look like?

- As we zoom in on the level curves of an arbitrary surface, they start to look more and more like equally spaced parallel lines

- For example: $f(x, y) = 2x^2 + y^2$



⇒ We can use tangent planes to approximate function values



- The **linear approximation** of f at (a, b) is

- Compare to equation for tangent plane above: use $x_0 = a, y_0 = b, z_0 = f(a, b)$

Example 3. Find the linear approximation of $f(x, y) = xe^{xy}$ at $(1, 0)$. Use it to approximate $f(1.1, -0.2)$.

Example 4. Here is the wind-chill index function $W(T, v)$ we have seen in previous lessons:

		Wind speed (km/h)											
		5	10	15	20	25	30	40	50	60	70	80	
Actual temperature (°C)	$T \setminus v$	5	4	3	2	1	1	0	-1	-1	-2	-2	-3
	0	-2	-3	-4	-5	-6	-6	-7	-8	-9	-9	-10	
	-5	-7	-9	-11	-12	-12	-13	-14	-15	-16	-16	-17	
	-10	-13	-15	-17	-18	-19	-20	-21	-22	-23	-23	-24	
	-15	-19	-21	-23	-24	-25	-26	-27	-29	-30	-30	-31	
	-20	-24	-27	-29	-30	-32	-33	-34	-35	-36	-37	-38	
	-25	-30	-33	-35	-37	-38	-39	-41	-42	-43	-44	-45	
	-30	-36	-39	-41	-43	-44	-46	-48	-49	-50	-51	-52	
	-35	-41	-45	-48	-49	-51	-52	-54	-56	-57	-58	-60	
	-40	-47	-51	-54	-56	-57	-59	-61	-63	-64	-65	-67	

Once upon a time, we estimated $W_T(-15, 40) \approx 1.3$. In a similar fashion, we can estimate $W_v(-15, 40) \approx -0.15$. Find the linear approximation of $W(T, v)$ at $(-15, 40)$. Use it to approximate $W(-12, 45)$.

- Why bother with linear approximations?
 - Desert island
 - More importantly: **linear functions** (functions of the form $f(x, y) = ax + by$) are much easier to deal with than other types of functions
 - ⇒ Linear approximations form the basis of many algorithms for complex problems
- Disclaimer: equations for tangent planes and linear approximations above do not necessarily apply when the partial derivatives of f are not continuous

3 Differentials

- Suppose we want to find the difference between $f(x, y)$ and $f(a, b)$: $f(x, y) - f(a, b)$
- Recall that the linear approximation of f at (a, b) is

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Letting $dx = x - a$ and $dy = y - b$ and rearranging terms, we get

- The **differential** of f is

- So we can approximate the difference between $f(x, y)$ and $f(a, b)$ by computing the differential df

Example 5. Let $f(x, y) = x^2 + 3xy - y^2$.

- Find the differential df .
- Use the differential to estimate the change in f when x changes from 2 to 2.05 and y changes from 3 to 2.96.
- Compare your approximation with the actual change.